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Maximum principles and the method of upper and lower solutions for time-periodic problems of the telegraph equations ☆

Yongxiang Li

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China

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Abstract

This paper deals with the existence of bounded time-periodic solutions for the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t = F(t, x, u), \quad (t, x) \in \mathbb{R}^2,$$

where $c > 0$ is a constant, $F \in C(\mathbb{R}^3, \mathbb{R})$ is 2π-periodic in t. We build a maximum principle for the time-periodic solutions of the corresponding linear telegraph equation. Using this maximum principle, we develop a method of the upper and lower solutions for the time-periodic problem of the nonlinear telegraph equation and obtain some existence and uniqueness results.
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Keywords: Telegraph equation; Time-periodic solution; Maximum principle; Upper and lower solutions

1. Introduction

In this paper we deal with the existence of time-periodic solutions for the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t = F(t, x, u), \quad (t, x) \in \mathbb{R}^2. \tag{1.1}$$

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E-mail address: liyx@nwnu.edu.cn.

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Because of its important physical background, the existence of time-periodic solutions of the telegraph equations with various boundary condition for space variable x has been studied by many authors, see [1–13] and the references therein. We consider the existence of the solutions $u(t,x)$ of Eq. (1.1), which satisfy the time-periodic condition

$$u(t+2\pi,x)=u(t,x), \quad (t,x)\in\mathbb{R}^2, \tag{1.2}$$

and are bounded on $\mathbb{R}$ with respect to space variable x. We term such solutions the bounded time-periodic solutions. Our discussion is based on a new maximum principle for time-periodic solutions of the linear telegraph equation

$$u_{tt}-u_{xx}+cu_t+a(t,x)u=h(t,x), \quad (t,x)\in\mathbb{R}^2, \tag{1.3}$$

where $a,h\in L^\infty(\mathbb{R}^2)$ and a satisfies certain conditions.

Maximum principles have many applications in the theory of differential equations. Specially, if a maximum principle holds for a linear differential equation, the method of lower and upper solutions is applicable to the corresponding nonlinear differential equation, see [14]. The first maximum principle for linear telegraph equations was built by Ortega and Robles-Pérez in [10]. They proved that the maximum principle for the double 2π-periodic solutions of the linear telegraph equation

$$u_{tt}-u_{xx}+cu_t+\lambda u=h(t,x), \quad (t,x)\in\mathbb{R}^2, \tag{1.4}$$

holds if and only if $\lambda\in(0,\nu(c)]$, where $\nu(c)\in(\frac{c^2}{4},\frac{c^2}{4}+\frac{1}{4}]$ is a constant which cannot be concretely determined. This maximum principle on the torus $\mathbb{T}^2$ (here, $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ denotes the unit circle) was used in [10] to develop a method of upper and lower solutions for the double periodic solutions of the nonlinear telegraph equation (1.1) when the function $u\mapsto F(t,x,u)+\nu(c)u$ is monotonically nondecreasing.

Afterwards in [15], Mawhin, Ortega and Robles-Pérez built a maximum principle for the solutions $u(t,x)$ of Eq. (1.4) which are 2π-periodic with respect to x and bounded on $\mathbb{R}$ with respect to t (namely, $u\in L^\infty(\mathbb{R}\times\mathbb{T})$) when $\lambda\in(0,\frac{c^2}{4}]$. In [15] a similar method of upper and lower solutions for the solutions of Eq. (1.1) in $L^\infty(\mathbb{R}\times\mathbb{T})$ was developed when the function $u\mapsto F(t,x,u)+\frac{c^2}{4}u$ is monotonically nondecreasing. Lately, these authors in [16] have extended their results in [15] to the telegraph equations in space dimensions two or three.

The purpose of this paper is to build a maximum principle for the bounded 2π-time-periodic solutions (namely, the solutions $u\in L^\infty(\mathbb{T}\times\mathbb{R})$) of linear equation (1.3), and use such a maximum principle to develop a method of upper and lower solutions for the bounded 2π-time-periodic solutions of the nonlinear equation (1.1). In Section 2, we first study the 2π-time-periodic problem for the linear telegraph equation with the form of

$$u_{tt}-u_{xx}+cu_t+\frac{c^2}{4}u=h(t,x), \quad (t,x)\in\mathbb{R}^2. \tag{1.5}$$

We use the distribution and fundamental solution theory to construct the Green function $G(t,x)$ of 2π-time-periodic solutions of Eq. (1.5). By the property of the Green function, we prove that for every $h\in L^\infty(\mathbb{T}\times\mathbb{R})$, Eq. (1.5) has a unique weak 2π-time-periodic solution $u(t,x)$, and $u\in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ satisfies the maximum principle: if $h(t,x)\geqslant 0$ a.e. in $\mathbb{T}\times\mathbb{R}$, the solution $u(t,x)\geqslant 0$ on $\mathbb{T}\times\mathbb{R}$. Then by the disturbance theory of positive operator, we obtain a maximum principle for the 2π-time-periodic solutions of the linear equation (1.3) when $\varepsilon\leqslant a(t,x)\leqslant\frac{c^2}{4}$ ($\varepsilon>0$). In Section 3, we use this maximum principle to develop a method of upper and lower solutions for bounded 2π-time-periodic solutions of the nonlinear equation (1.1) when the function

$u \mapsto F(t,x,u) + \frac{c^2}{4}u$ is monotonically nondecreasing. Moreover, we use this maximum principle to construct the upper and lower solutions for the 2π-time-periodic solutions of Eq. (1.1) and obtain some existence and uniqueness results.

Our results are applied to the forced dissipative sine-Gordon equation

$$u_{tt} - u_{xx} + cu_t + a(t,x)\sin u = h(t,x). \tag{1.6}$$

We show that Eq. (1.6) has infinitely many bounded 2π-time-periodic solutions when $a, h \in C(\mathbb{T}\times\mathbb{R}) \cap L^\infty(\mathbb{T}\times\mathbb{R})$ satisfy the conditions that $\varepsilon \leqslant a(t,x) \leqslant \frac{c^2}{4}$ and $|h(t,x)| \leqslant a(t,x)$.

2. Preliminaries on linear telegraph equations and maximum principles

Throughout this paper, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denotes the unit circle, and functions $u(t,x)$ defined on $\mathbb{R}^2$ which are 2π-periodic with respect to t will be identified to functions defined on $\mathbb{T}\times\mathbb{R}$. In particular, the notations

$$L^1(\mathbb{T}\times\mathbb{R}), \qquad L^\infty(\mathbb{T}\times\mathbb{R}), \qquad C(\mathbb{T}\times\mathbb{R}), \qquad \mathcal{D}(\mathbb{T}\times\mathbb{R}) = C_0^\infty(\mathbb{T}\times\mathbb{R}), \qquad \ldots$$

denote the spaces of t-periodic functions with the indicated degree regularity. $\mathcal{D}'(\mathbb{T}\times\mathbb{R})$ denotes the space of distribution on $\mathbb{T}\times\mathbb{R}$, and $\mathcal{D}'(\mathbb{R}^2)$ is the space of distribution on $\mathbb{R}^2$.

In this section, we first consider the linear time-periodic problem with the special form of

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = h(t,x) \quad \text{in } \mathcal{D}'(\mathbb{T}\times\mathbb{R}), \tag{2.1}$$

where $c > 0$ is a constant, and $h \in L_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R})$. We follow the ideas that [10] deals with the double periodic problem to discuss Eq. (2.1).

To be convenient, we define the differential operator $\mathcal{L}$ acting on functions on $\mathbb{T}\times\mathbb{R}$ by

$$\mathcal{L}u = u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u.$$

The formal adjoint operator of $\mathcal{L}$ is

$$\mathcal{L}^*u = u_{tt} - u_{xx} - cu_t + \frac{c^2}{4}u.$$

By a solution of Eq. (2.1) we understand a function $u \in L_{\mathrm{loc}}(\mathbb{T}\times\mathbb{R})$ satisfying the equation in the distribution sense (weak solution), that is

$$\int\limits_{\mathbb{T}\times\mathbb{R}} u\mathcal{L}^*\psi\,dt\,dx = \int\limits_{\mathbb{T}\times\mathbb{R}} h\psi\,dt\,dx, \quad \forall\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{R}).$$

We seek the fundamental solution of the operator $\mathcal{L}$ in $\mathcal{D}'(\mathbb{T}\times\mathbb{R})$ and express the solution of Eq. (2.1) by integration. Let δ be Dirac distribution in $\mathcal{D}'(\mathbb{R}^2)$, more precisely

$$\langle\delta,\phi\rangle = \phi(0,0), \quad \forall\phi \in \mathcal{D}(\mathbb{R}^2),$$

and $\bar{\delta}$ be Dirac distribution in $\mathcal{D}'(\mathbb{T}\times\mathbb{R})$.

It is well known that the function

$$U_0(t,x) = \begin{cases} \frac{1}{2}, & |x| < t, \\ 0, & |x| \geqslant t, \end{cases}$$

is the fundamental solution of the wave operator $\Box u = u_{tt} - u_{xx}$ in $\mathcal{D}'(\mathbb{R}^2)$, namely

$$U_{0tt} - U_{0xx} = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

From this we easily see that the fundamental solution of operator $\mathcal{L}$ in $\mathcal{D}'(\mathbb{R}^2)$ is given by

$$U(t,x) = e^{-(c/2)t} U_0(t,x), \tag{2.2}$$

that is

$$U_{tt} - U_{xx} + cU_t + \frac{c^2}{4} U = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^2). \tag{2.3}$$

To obtain the fundamental solution of operator $\mathcal{L}$ in $\mathcal{D}'(\mathbb{T} \times \mathbb{R})$, we construct the t-periodic function

$$G(t,x) = \sum_{n \in \mathbb{Z}} U(t + 2n\pi, x), \quad (t,x) \in \mathbb{R}^2. \tag{2.4}$$

By the Weierstrass test, it is easy to show that the series in (2.4) is uniformly convergent on bounded subset of $\mathbb{R}^2$, and therefore $G \in L^\infty_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R})$.

Lemma 2.1. *The function G defined by* (2.4) *is the fundamental solution of operator $\mathcal{L}$ in $\mathcal{D}'(\mathbb{T} \times \mathbb{R})$, namely*

$$G_{tt} - G_{xx} + cG_t + \frac{c^2}{4} G = \bar{\delta} \quad \text{in } \mathcal{D}'(\mathbb{T} \times \mathbb{R}). \tag{2.5}$$

Proof. For every $\phi \in \mathcal{D}(\mathbb{T} \times \mathbb{R})$, by (2.3) we have

$$\int_{\mathbb{R}^2} U \mathcal{L}^* \phi \, dt \, dx = \phi(0,0),$$

from which and (2.4) it follows that

$$\int_{\mathbb{R}^2} G \mathcal{L}^* \phi \, dt \, dx = \sum_{n \in \mathbb{Z}} \phi(2n\pi, 0). \tag{2.6}$$

Choose a function $\varphi_0 \in \mathcal{D}(\mathbb{R}) = C_0^\infty(\mathbb{R})$ such that

$$\operatorname{supp} \varphi_0 \subset \left[-\frac{3\pi}{2}, \frac{3\pi}{2} \right], \qquad \sum_{n \in \mathbb{Z}} \varphi_0(t + 2n\pi) \equiv 1. \tag{2.7}$$

Such a function can be constructed as follows:

Choosing $\bar{\varphi} \in \mathcal{D}(\mathbb{R})$ such that $0 \leqslant \bar{\varphi}(t) \leqslant 1$ and

$$\bar{\varphi}(t) = \begin{cases} 1, & t \in [-\pi, \pi], \\ 0, & t \notin [-3\pi/2, 3\pi/2], \end{cases}$$

set

$$\varphi_0(t) = \bar{\varphi}(t) \Big/ \left(\sum_{n \in \mathbb{Z}} \bar{\varphi}(t + 2n\pi) \right),$$

then $\varphi_0 \in \mathcal{D}(\mathbb{R})$ satisfies (2.7).

By (2.7) we can easily verify that, for every $h \in L_{\mathrm{loc}}(\mathbb{T} \times \mathbb{R})$ and $\alpha \in \mathbb{N}^2$ with $|\alpha| > 0$, φ_0 has the following properties:

(i) $$\int_{\mathbb{R}^2} h\varphi_0 \, dt \, dx = \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}\times\mathbb{R}} h\varphi_0(t+2n\pi) \, dt \, dx = \int_{\mathbb{T}\times\mathbb{R}} h \, dt \, dx;$$

(ii) $$\int_{\mathbb{R}^2} h\partial^\alpha\varphi_0 \, dt \, dx = \sum_{n\in\mathbb{Z}} \int_{\mathbb{T}\times\mathbb{R}} h\partial^\alpha\varphi_0(t+2n\pi) \, dt \, dx = 0.$$

Now for every $\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{R})$, since $\psi\varphi_0 \in \mathcal{D}(\mathbb{R}^2)$, by the properties (i)–(ii) and (2.6), we have

$$\begin{aligned}\int_{\mathbb{T}\times\mathbb{R}} G\mathcal{L}^*\psi \, dt \, dx &= \int_{\mathbb{R}^2} (G\mathcal{L}^*\psi)\varphi_0 \, dt \, dx = \int_{\mathbb{R}^2} G\mathcal{L}^*(\psi\varphi_0) \, dt \, dx \\ &= \sum_{n\in\mathbb{Z}} (\psi\varphi_0)(2n\pi, 0) = \psi(0,0) = \langle\bar{\delta}, \psi\rangle.\end{aligned}$$

In consequence, we obtain that

$$\langle\mathcal{L}G, \psi\rangle = \langle G, \mathcal{L}^*\psi\rangle = \langle\bar{\delta}, \psi\rangle, \quad \forall\psi \in \mathcal{D}(\mathbb{T}\times\mathbb{R}),$$

this means that (2.5) holds. □

We seek to obtain the precise expression of G. Let

$$\mathcal{D}_k = \{(t,x) \in \mathbb{R}^2\colon\ |x| + 2k\pi < t < |x| + 2k\pi + 2\pi\},$$

and $\mathcal{D} = \bigcup_{k=-\infty}^{\infty} \mathcal{D}_k$. Then the complementary set of $\mathcal{D}$,

$$\mathcal{C} = \mathbb{R}^2 \setminus \mathcal{D} = \{(t,x) \in \mathbb{R}^2\colon\ t = |x| + 2k\pi,\ k = 0, \pm1, \pm2, \ldots\},$$

has measure zero. By t-periodicity, the value of G on $\mathcal{D}_0$ determines the value of it on whole set $\mathcal{D}$. For every $(t,x) \in \mathcal{D}_0$, noting that $U(t,x)$ vanishes on the outside of the open cone $V = \{(t,x) \in \mathbb{R}^2\colon\ t > 0,\ |x| < t\}$, by (2.4) we obtain that

$$\begin{aligned}G(t,x) &= \sum_{(t+2n\pi,x)\in V} U(t+2n\pi, x) \\ &= \sum_{n=0}^{\infty} \frac{1}{2} e^{-(c/2)t} e^{-nc\pi} = \frac{e^{-(c/2)t}}{2(1-e^{-c\pi})}.\end{aligned}$$

Hence, we have

Lemma 2.2. *The fundamental solution G is precisely expressed by*

$$G(t,x) = \frac{e^{-(c/2)t}}{2(1-e^{-c\pi})}, \quad (t,x) \in \mathcal{D}_0. \tag{2.8}$$

By (2.8), $G(t,x)$ is continuous on every $\mathcal{D}_k$. Furthermore, it is easy to see that

$$G \in L^\infty(\mathbb{T}\times\mathbb{R}) \cap L^1(\mathbb{T}\times\mathbb{R}) \tag{2.9}$$

with the L^1-norm

$$\|G\|_{L^1} = \int_{-\infty}^{\infty}\int_0^{2\pi} G(t,x)\,dt\,dx = 2\int_0^{\infty}\int_0^{2\pi} G(t,x)\,dt\,dx$$
$$= 2\int_0^{\infty}\int_x^{x+2\pi} G(t,x)\,dt\,dx$$
$$= \frac{1}{1-e^{-c\pi}}\int_0^{\infty}\int_x^{x+2\pi} e^{-(c/2)t}\,dt\,dx = \frac{4}{c^2}$$

and essential supremum

$$\overline{G} := \operatorname{ess\,sup} G(t,x) = \frac{1}{2(1-e^{-c\pi})}.$$

Remark 2.1. The proofs of the above Lemmas 2.1 and 2.2 are similar to those of Lemmas 5.1 and 5.2 in [10]. But the working spaces are different, and in our proofs some detailed computations concerning φ_0 and G are given and the argumentations are more explicit.

Now we consider the time-periodic problem (2.1). For every $h \in L^1(\mathbb{T}\times\mathbb{R}) \cup L^\infty(\mathbb{T}\times\mathbb{R})$, by Lemma 2.1 and (2.9), the convolution

$$u(t,x) = G * h = \int_{\mathbb{T}\times\mathbb{R}} G(t-s, x-y)h(s,y)\,ds\,dy \tag{2.10}$$

is a unique weak solution of Eq. (2.1). By the continuity of G on $\mathcal{D}$ and the dominated convergence theorem, $u \in C(\mathbb{T}\times\mathbb{R})$. Moreover, if $h(t,x) \geqslant 0$ a.e. in $\mathbb{T}\times\mathbb{R}$, by the positivity of G and (2.10), the solution $u(t,x) \geqslant 0$ on $\mathbb{T}\times\mathbb{R}$, that is, Eq. (2.1) satisfies the maximum principle.

Let $W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ denote the Banach space of functions $u \in L^\infty(\mathbb{T}\times\mathbb{R})$ which are Lipschitz-continuous, with the norm

$$\|u\|_{W^{1,\infty}} = \|u\|_{L^\infty} + [u]_{\mathrm{Lip}},$$

where $[u]_{\mathrm{Lip}}$ is the best Lipschitz constant of u.

Theorem 2.1. *For every $h \in L^\infty(\mathbb{T}\times\mathbb{R})$, Eq.* (2.1) *has a unique solution $u(t,x)$ which belongs to $W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ and satisfies the estimates*

$$\|u\|_{L^\infty} \leqslant \frac{4}{c^2}\|h\|_{L^\infty}, \qquad \|u\|_{W^{1,\infty}} \leqslant C\|h\|_{L^\infty},$$

where C is a positive constant. Moreover, if $h(t,x) \geqslant 0$ a.e. in $\mathbb{T}\times\mathbb{R}$, the solution $u(t,x) \geqslant 0$ on all $\mathbb{T}\times\mathbb{R}$.

Proof. For every $h \in L^\infty(\mathbb{T}\times\mathbb{R})$, Eq. (2.1) has a unique weak solution $u \in C(\mathbb{T}\times\mathbb{R})$ given by (2.10). By (2.10) u satisfies that

$$\|u\|_{L^\infty} \leqslant \|G\|_{L^1}\|h\|_{L^\infty} = \frac{4}{c^2}\|h\|_{L^\infty},$$

and can be expressed as

$$
\begin{aligned}
u(t,x) &= \int\limits_{\mathbb{T}\times\mathbb{R}} G(t-s,x-y)h(s,y)\,ds\,dy \\
&= \int\limits_{\mathbb{T}\times\mathbb{R}} \left[\sum_{n\in\mathbb{Z}} U(t-s+2n\pi,x-y)h(s,y)\right] ds\,dy \\
&= \int\limits_{\mathbb{R}^2} U(t-s,x-y)h(s,y)\,ds\,dy.
\end{aligned}
$$

Given $\varepsilon>0$, we denote by $\chi_\varepsilon(t,x)$ the characteristic function of the set

$$\mathcal{B}_\varepsilon = \bigl\{(t,x)\in\mathbb{R}^2\colon\ |x|-\varepsilon<t<|x|-\varepsilon\bigr\}.$$

Let $\chi_V(t,x)$ denote the characteristic function of open cone V.

Given $k=(k_1,k_2)\in\mathbb{R}^2$ with $|k|=|k_1|+|k_2|\leqslant 1$, considering two case of $(t,x)\in\mathcal{B}_\varepsilon$ and $(t,x)\notin\mathcal{B}_\varepsilon$, we easily prove that

$$\bigl|U(t+k_1,x+k_2)-U(t,x)\bigr|\leqslant C_1\bigl[|k|\chi_V(t,x)+\chi_{|k|}(t,x)\bigr]e^{-(c/2)t},$$

for all $(t,x)\in\mathbb{R}^2$, here and after, $C_1,C_2,C_3,\ldots$ denote some positive constants. Since

$$\int\limits_{\mathbb{R}^2}\chi_V(t,x)e^{-(c/2)t}\,dt\,dx=\frac{8}{c^2},\qquad \int\limits_{\mathbb{R}^2}\chi_{|k|}(t,x)e^{-(c/2)t}\,dt\,dx\leqslant C_2|k|,$$

we obtain that

$$\int\limits_{\mathbb{R}^2}\bigl|U(t+k_1,x+k_2)-U(t,x)\bigr|\,dt\,dx\leqslant C_3|k|.$$

Consequently,

$$
\begin{aligned}
&\bigl|u(t+k_1,x+k_2)-u(t,x)\bigr| \\
&\quad\leqslant \int\limits_{\mathbb{R}^2}\bigl|U(t+k_1-s,x+k_2-y)-U(t-s,x-y)\bigr|\cdot\bigl|h(s,y)\bigr|\,ds\,dy \\
&\quad\leqslant \int\limits_{\mathbb{R}^2}\bigl|U(s+k_1,y+k_2)-U(s,y)\bigr|\,ds\,dy\cdot\|h\|_{L^\infty} \\
&\quad\leqslant C_3\|h\|_{L^\infty}|k|.
\end{aligned}
$$

This implies that $[u]_{\mathrm{Lip}}\leqslant C_3\|h\|_{L^\infty}$, and thus $\|u\|_{W^{1,\infty}}\leqslant C\|h\|_{L^\infty}$. $\square$

Now we consider the time-periodic problem of the linear telegraph equation with variable coefficients

$$u_{tt}-u_{xx}+cu_t+a(t,x)u=h(t,x)\quad\text{in }\mathcal{D}'(\mathbb{T}\times\mathbb{R}),\tag{2.11}$$

where $a\in L^\infty(\mathbb{T}\times\mathbb{R})$ satisfies the assumption:

(P) $0\leqslant a(t,x)\leqslant\frac{c^2}{4}$ a.e. in $\mathbb{T}\times\mathbb{R}$, and $\underline{a}=\operatorname{ess\,inf} a(t,x)>0$.

Theorem 2.2. *Assume $a \in L^\infty(\mathbb{T}\times\mathbb{R})$ satisfies* (P). *Then for every $h \in L^\infty(\mathbb{T}\times\mathbb{R})$, Eq.* (2.11) *has a unique solution $u := Ph$, this solution belongs to $W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ and $P : L^\infty(\mathbb{T}\times\mathbb{R}) \to W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ is a bounded linear operator. Moreover, the maximum principle holds*: *if $h(t,x) \geqslant 0$ a.e. in $\mathbb{T}\times\mathbb{R}$, the solution $u(t,x) \geqslant 0$ on all $\mathbb{T}\times\mathbb{R}$.*

Proof. Let E denote the Banach space $L^\infty(T\times\mathbb{R})$, whose essential supremum norm is simply denoted by $\|u\|$. E is also an ordered Banach space with the positive cone $K = \{h \in E\colon\ h(t,x) \geqslant 0 \text{ a.e. in } T\times\mathbb{R}\}$.

We use the same method with the proof of Lemma 2 of [13] to prove Theorem 2.2. In order to do this, we define a positive linear operator $\mathcal{R} : E \to E$ by

$$(\mathcal{R}h)(t,x) = \int\limits_{\mathbb{T}\times\mathbb{R}} G(t-s,x-y)h(s,y)\,ds\,dy, \tag{2.12}$$

which is the inverse operator of $\mathcal{L}$. By Theorem 2.1, $\mathcal{R} : E \to W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ is bounded, and its norm in $\mathcal{L}(E,E)$ satisfies $\|\mathcal{R}\| \leqslant \frac{4}{c^2}$. Now we rewrite Eq. (2.11) in form of the operator equation in E,

$$(I - \mathcal{R}\circ B)u = \mathcal{R}h, \tag{2.13}$$

where I is the identity operator in E, and $B : E \to E$ is defined by

$$(Bu)(t,x) = \left(\frac{c^2}{4} - a(t,x)\right)u(t,x), \quad \forall u \in E, \tag{2.14}$$

which is a positive linear operator. For every $v \in E$, because $\mathcal{R}\circ B$ is a positive linear operator in E, we have

$$\begin{aligned}
\bigl|(\mathcal{R}\circ B)(v)\bigr| &\leqslant (\mathcal{R}\circ B)\bigl(\|v\|\cdot 1\bigr) = \|v\|\cdot(\mathcal{R}\circ B)(1) \\
&= \|v\|\cdot\mathcal{R}\left(\frac{c^2}{4} - a(t,x)\right) \\
&\leqslant \|v\|\cdot\left(\frac{c^2}{4} - \underline{a}\right)\mathcal{R}(1) = \left(1 - \frac{4}{c^2}\underline{a}\right)\|v\|.
\end{aligned}$$

This shows that the norm of $\mathcal{R}\circ B$ in $\mathcal{L}(E,E)$ satisfies

$$\|\mathcal{R}\circ B\| \leqslant 1 - \frac{4}{c^2}\underline{a} < 1. \tag{2.15}$$

Thus, $I - \mathcal{R}\circ B$ has the bounded inverse operator given by the series

$$(I - \mathcal{R}\circ B)^{-1} = \sum_{n=0}^{\infty}(\mathcal{R}\circ B)^n.$$

Consequently, Eq. (2.13), equivalently Eq. (2.11), has a unique solution

$$u = (I - \mathcal{R}\circ B)^{-1}(\mathcal{R}h) := Ph,$$

where

$$P = (I - \mathcal{R}\circ B)^{-1}\mathcal{R} = \sum_{n=0}^{\infty}(\mathcal{R}\circ B)^n\mathcal{R}. \tag{2.16}$$

It is clear that $P \in \mathcal{L}(E, E)$ is a positive operator with the norm estimate $\|P\| \leqslant 1/\underline{a}$. From (2.16) we see that P can be expressed by

$$P = \mathcal{R}\left(I + B \circ \mathcal{R} + \sum_{n=1}^{\infty} B(\mathcal{R} \circ B)^n \mathcal{R}\right) = \mathcal{R} \circ Q,$$

where

$$Q = I + B \circ \mathcal{R} + \sum_{n=1}^{\infty} B(\mathcal{R} \circ B)^n \mathcal{R}$$

is in $\mathcal{L}(E, E)$. By the boundedness of $\mathcal{R} : E \to W^{1,\infty}(\mathbb{T} \times \mathbb{R})$, $P = \mathcal{R} \circ Q$ maps E into $W^{1,\infty}(\mathbb{T} \times \mathbb{R})$ and is bounded.

This completes the proof of Theorem 2.2. □

Remark 2.2. For $\lambda \in (0, \frac{c^2}{4}]$, in Theorem 2.2 choosing $a(t, x) \equiv \lambda$, we see that the differential operator

$$\mathcal{L}_\lambda u = u_{tt} - u_{xx} + cu_t + \lambda u$$

satisfies the maximum principle for $u \in L^\infty(\mathbb{T} \times \mathbb{R})$. Similarly to the case that $u \in L^\infty(\mathbb{R} \times \mathbb{T})$ in [15], the constant $\frac{c^2}{4}$ is optimal in the maximum principle, and the maximum principle is not strong.

3. The method of upper and lower solutions

In this section, we use a method of upper and lower solutions to discuss the existence of the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t = F(t, x, u) \quad \text{in } \mathcal{D}'(\mathbb{T} \times \mathbb{R}), \tag{3.1}$$

where $F : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$ is continuous.

Definition 3.1. Let $\mathcal{L}_0$ be the differential operator defined by

$$\mathcal{L}_0 u = u_{tt} - u_{xx} + cu_t.$$

If a function $v \in L^\infty(\mathbb{T} \times \mathbb{R})$, such that $\mathcal{L}_0 v \in L^\infty(\mathbb{T} \times \mathbb{R})$ in the sense of distribution $\mathcal{D}'(\mathbb{T} \times \mathbb{R})$, and satisfies

$$\mathcal{L}_0 v \leqslant F(t, x, v) \quad \text{a.e. } (t, x) \in \mathbb{T} \times \mathbb{R},$$

we call it a lower solution of Eq. (3.1). If the inequality is inverse, we call it an upper solution of Eq. (3.1).

Theorem 3.1. *Let Eq.* (3.1) *have a lower solution v_0 and an upper solution w_0 with $v_0 \leqslant w_0$. Assume that $F(t, x, u) \in C(T \times \mathbb{R}^2, \mathbb{R})$ satisfies the monotonicity condition*:

(H1) $$F(t,x,\eta_2) - F(t,x,\eta_1) \geqslant -\frac{c^2}{4}(\eta_2 - \eta_1),$$
$$\forall (t,x) \in \mathbb{T}, \text{ and } v(t,x) \leqslant \eta_1 \leqslant \eta_2 \leqslant w(t,x).$$

Then Eq. (3.1) *has at least one solution* $u \in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ *between* v_0 *and* w_0.

Proof. We choose the ordered Banach space $E = L^\infty(T\times\mathbb{R})$, which positive cone K is normal. Let $D = [v_0, w_0]$ be the order-interval in E. Set

$$\mathcal{F}(u)(t,x) := F\big(t,x,u(t,x)\big) + \frac{c^2}{4}u(t,x), \quad u \in D,$$

then $\mathcal{F}: D \to E$ is continuous. By the assumption (H1), $\mathcal{F}: D \to E$ is an increasing mapping. By Theorem 2.1, differential operator $\mathcal{L}$ has the positively inverse operator $\mathcal{R} \in \mathcal{L}(E,E)$. Since Eq. (3.1) can be rewritten in the form of

$$\mathcal{L}u = F(t,x,u) + \frac{c^2}{4}u \quad \text{in } \mathcal{D}'(\mathbb{T}\times\mathbb{R}),$$

by the definition of $\mathcal{R}$, the solution of Eq. (3.1) in D is equivalent to the fixed-point of the operator $A = \mathcal{R}\circ\mathcal{F}$. Clearly, $A: D \to E$ is a continuously increasing operator. Let $h_1 = \mathcal{L}v_0$, $h_2 = \mathcal{L}w_0$. By the definition of the lower and upper solutions, $h_1, h_2 \in L^\infty(\mathbb{T}\times\mathbb{R})$, and

$$h_1 \leqslant \mathcal{F}(v_0), \qquad \mathcal{F}(w_0) \leqslant h_2,$$

which implies that: $v_0 \leqslant Av_0$, $Aw_0 \leqslant w_0$. Combining this fact with the increasing property of A in D, we see that A maps D into itself.

Now, we define two sequences $\{v_n\}$ and $\{w_n\}$ in D by the iterative scheme

$$v_n = Av_{n-1}, \quad w_n = Aw_{n-1}, \quad n = 1, 2, \ldots. \tag{3.2}$$

Then from the monotonicity of A, it follows that

$$v_0 \leqslant v_1 \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant w_n \leqslant \cdots \leqslant w_1 \leqslant w_0. \tag{3.3}$$

Let

$$\underline{u}(t,x) = \lim_{n\to\infty} v_n(t,x), \qquad \bar{u}(t,x) = \lim_{n\to\infty} w_n(t,x), \tag{3.4}$$

then $\underline{u}, \bar{u} \in E$, $v_0 \leqslant \underline{u} \leqslant \bar{u} \leqslant w_0$. Letting $n\to\infty$ in (3.2), since $\mathcal{F}(v_0) \leqslant \mathcal{F}(v_n) \leqslant \mathcal{F}(w_n) \leqslant \mathcal{F}(w_0)$ $(n = 1, 2, \ldots)$, by the expression (2.12) of $\mathcal{R}$ and the dominated convergence theorem, we obtain that

$$\underline{u} = A\underline{u}, \qquad \bar{u} = A\bar{u}. \tag{3.5}$$

Therefore, $\underline{u}$ and $\bar{u}$ are two solutions of Eq. (3.1) in D. Since $\mathcal{R}$ maps E into $W^{1,\infty}(\mathbb{T}\times\mathbb{R})$, it follows that $A(D) = \mathcal{R}(\mathcal{F}(D)) \subset W^{1,\infty}(\mathbb{T}\times\mathbb{R})$. Thus $\underline{u}, \bar{u} \in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$. Moreover, we easily see that $\underline{u}$ and $\bar{u}$ are respectively the minimal solution and maximal solution of Eq. (3.1) in $[v_0, w_0]$.

The proof of Theorem 3.1 is completed. □

Strengthening the conditions in Theorem 3.1, we have the following unique existence result of Eq. (3.1) in $[v_0, w_0]$.

Theorem 3.2. *Let Eq.* (3.1) *have a lower solution* v_0 *and an upper solution* w_0 *with* $v_0 \leqslant w_0$. *Assume that* $F(t,x,u) \in C(T\times\mathbb{R}^2, \mathbb{R})$ *satisfies* (H1) *and the following condition*:

(H2) *There exists* $a \in L^\infty(\mathbb{T}\times\mathbb{R})$ *satisfying condition* (P) *such that*

$$F(t,x,\eta_2) - F(t,x,\eta_1) \leqslant -a(t,x)(\eta_2-\eta_1),$$

$$\forall (t,x)\in\mathbb{T}, \text{ and } v(t,x)\leqslant \eta_1\leqslant\eta_2\leqslant w(t,x).$$

Then Eq. (3.1) *has a unique solution between* v_0 *and* w_0*, moreover this solution belongs to* $W^{1,\infty}(\mathbb{T}\times\mathbb{R})$.

Proof. By Theorem 3.1, Eq. (3.1) has a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ in $[v_0, w_0]$. By the proof of Theorem 3.1, (3.2)–(3.5) are valid. By (3.2), (3.3) and assumptions (H1) and (H2), we have

$$\begin{aligned} 0 \leqslant w_n - v_n &= A(w_{n-1}) - A(v_{n-1}) \\ &= \mathcal{R}\big(\mathcal{F}(w_{n-1}) - \mathcal{F}(v_{n-1})\big) \\ &\leqslant \mathcal{R}\left(\left(\frac{c^2}{4} - a\right)(w_{n-1}-v_{n-1})\right) \\ &\leqslant (\mathcal{R}\circ B)(w_{n-1}-v_{n-1}), \end{aligned}$$

where $B\in\mathcal{L}(E,E)$ is the positive linear operator defined by (2.14). Repeatedly using this inequality, we have

$$0\leqslant w_n - v_n \leqslant (\mathcal{R}\circ B)^n(w_0-v_0), \quad n=1,2,\ldots, \tag{3.6}$$

from which and (2.15) it follows that

$$\begin{aligned} \|\bar{u}-\underline{u}\| \leqslant \|w_n - v_n\| &\leqslant \big\|(\mathcal{R}\circ B)^n\big\|\cdot\|w_0-v_0\| \\ &\leqslant \left(1-\frac{4}{c^2}\underline{a}\right)^n \|w_0-v_0\| \to 0 \quad (n\to 0). \end{aligned}$$

Therefore, $\tilde{u} = \underline{u} = \bar{u}$ is the unique solution of Eq. (3.1) in $[v_0, w_0]$. □

Theorem 3.3. *Let* $F\in C(\mathbb{T}\times\mathbb{R}^2,\mathbb{R})$ *have the partial derivative* $F_\eta(t,x,\eta)$ *and* $F(t,x,0)\in L^\infty(\mathbb{T}\times\mathbb{R})$*. If there exists* $a\in L^\infty(\mathbb{T}\times\mathbb{R})$ *satisfying* (P) *such that*

$$-\frac{c^2}{4}\leqslant F_\eta(t,x,\eta)\leqslant -a(t,x), \quad \forall (t,x,\eta)\in\mathbb{T}\times\mathbb{R}^2, \tag{3.7}$$

then Eq. (3.1) *has a unique bounded solution* $u\in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$.

Proof. Let $h(t,x) = |F(t,x,0)|$, and let w be the unique bounded solution of the time-periodic problem (2.11). Since $h\in K$, by Theorem 2.2, $w\in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ is nonnegative. By the second inequality of (3.7),

$$F(t,x,w) - F(t,x,0)\leqslant -a(t,x)w, \quad (t,x)\in\mathbb{T}\times\mathbb{R},$$

$$F(t,x,0) - F(t,x,-w)\leqslant -a(t,x)w, \quad (t,x)\in\mathbb{T}\times\mathbb{R}.$$

This implies that

$$\mathcal{L}_0 w = -a(t,x)w + h(t,x)\geqslant F(t,x,w), \quad (t,x)\in\mathbb{T}\times\mathbb{R},$$

$$\mathcal{L}_0(-w) = -a(t,x)(-w) - h(t,x)\leqslant F(t,x,-w), \quad (t,x)\in\mathbb{T}\times\mathbb{R}.$$

That is, $-w$ and w are respectively a lower solution and an upper solution of Eq. (3.1). By the first inequality of (3.7), $F(t,x,u)$ satisfies assumption (H1) on order-interval $[-w,w]$, and by Theorem 3.1, Eq. (3.1) has a solution $u_1 \in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ between $-w$ and w.

Let $u_2 \in L^\infty(\mathbb{T}\times\mathbb{R})$ be another solution of Eq. (3.1). Setting $u = u_2 - u_1$, then we have

$$\mathcal{L}u = \left(\frac{c^2}{4} + F_\eta(t,x,u_1+\theta u)\right)u \quad \text{in } \mathcal{D}'(\mathbb{T}\times\mathbb{R}),$$

with $\theta \in [0,1]$. Therefore

$$u = \mathcal{R}\left(\left(\frac{c^2}{4} + F_\eta(t,x,u_1+\theta u)\right)u\right),$$

from which it follows that

$$\begin{aligned}\|u\| &= \|\mathcal{R}\| \cdot \left\|\left(\frac{c^2}{4} + F_\eta(t,x,u_1+\theta u)\right)u\right\| \\ &\leqslant \frac{4}{c^2}\cdot\left(\frac{c^2}{4} - \underline{a}\right)\cdot\|u\| \\ &\leqslant \left(1 - \frac{4}{c^2}\underline{a}\right)\cdot\|u\| < \|u\|.\end{aligned}$$

Thus $\|u\| = 0$, namely $u_1 = u_2$. This means that Eq. (3.1) has only one bounded solution. □

Remark 3.1. In Theorem 3.3, if $F(t,x,0) \geqslant 0$, we can choose $v \equiv 0$ as a lower solution of Eq. (3.1). In this case, the unique bounded solution of Eq. (3.1) is nonnegative.

Example 3.1. Consider the forced dissipative sine-Gordon equation with variable coefficient

$$u_{tt} - u_{xx} + cu_t + a(t,x)\sin u = h(t,x), \tag{3.8}$$

where $a, h \in C(\mathbb{T}\times\mathbb{R}) \cap L^\infty(\mathbb{T}\times\mathbb{R})$ and a satisfies condition (P). When $|h(t,x)| \leqslant a(t,x)$ a.e. in $\mathbb{T}\times\mathbb{R}$, Eq. (3.8) has infinitely many bounded 2π-time-periodic solutions. For every $k \in \mathbb{Z}$, since $v_k = 2k\pi - \pi/2$ and $w_k = 2k\pi + \pi/2$ are respectively the lower solution and upper solution of the time-periodic problem of this equation, and $F(t,x,u) = h(t,x) - a(t,x)\sin u$ satisfies the assumption (H1) on $[v_k, w_k]$, by Theorem 3.1, Eq. (3.8) has a bounded 2π-time-periodic solution $u_k \in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ between v_k and w_k.

Example 3.2. We consider the superlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t + du^{2n+1} = h(t,x), \tag{3.9}$$

where $n \in \mathbb{N}$, $h \in C(\mathbb{T}\times\mathbb{R}) \cap L^\infty(\mathbb{T}\times\mathbb{R})$ with $\|h\|_{L^\infty} > 0$, $d > 0$ is a constant. Let $R = \sqrt[2n+1]{\|h\|_{L^\infty}/d}$. It is easy to verify that $v_0 = -R$ is a lower solution and $w_0 = R$ is an upper solution of the time-periodic problem of Eq. (3.9). If

$$0 < d \leqslant \left(\frac{c^2}{4(2n+1)}\right)^{2n+1} \Big/ \|h\|_{L^\infty}^{2n}, \tag{3.10}$$

then $F(t,x,u) = h(t,x) - du^{2n+1}$ satisfies the monotonicity condition (H1) on $[v_0, w_0]$. Hence, when the inequality (3.10) holds, by Theorem 3.1, Eq. (3.9) has a bounded 2π-time-periodic solution $u \in W^{1,\infty}(\mathbb{T}\times\mathbb{R})$ such that $\|u\| \leqslant R$.

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